UNIT - IV

LATTICES :

One of the important concepts in Mathematics is that of relation. special types of relations are equivalence relations, functions are ordering relations. This chapter is devoted to the study on ordering relations.

we introduce a lattice as a partially ordered set and then concentrate our study on Boolean algebra, which is a special type of lattices.

we recall that a relation R on a non-empty set A is a subset of $A \times A$. If $(x,y) \in R$, then the we write xRy. A relation R on A is said to be reflexive if aRa holds good for every element $a \in A$, symmetric if for all $a,b \in A$, whenever $(a,b) \in R$, Then $(b,a) \in R$.

The relation R is said to be antisymmetric if aRb and bRa both hold good, then $a=b$. (This means that if $a \neq b$ and $(a,b) \in R$, then $(b,a) \notin R$). R is said to be transitive if whenever aRb and bRc, Then aRc

A relation R is said to be an equivalence relation if R is reflexive symmetric and transitive.

A relation R is said to be an partial ordering on A if R is reflexive, antisymmetric and transitive.

If R is a partial ordering on A, then (A, R) is called a partially ordered set or a poset. usually we write $\leq$ or $\subseteq$ instead of R.

Example :

Let N be the set of all positive integers, let the relation $\leq$ be the usual order relation in N. That is, $m \leq n$ if and only if $n - m$ is a non-negative integer. one can easily prove that $(N, \leq)$ is a partially ordered set.

HASSE DIAGRAMS :

A partially ordered finite set $(A, \leq)$ can be graphically represented by a diagram called Hasse diagram of the post $(A, \leq)$. The elements of A are represented as a points in the plane such that if $a, b \in A$ such that $a \leq b$ and $a \neq b$ (i.e, $a < b$) then the point b is plotted above the point a.

The point b need not be exactly vertically above a; there may be a deviation to the left or right of the vertical line through a. If $a < b$ and there is no c in A such that $a < c$ and $c < b$, then a and b are connected by a line segment.

For example, the Hasse diagram of the poset $(P(X), \subseteq)$ is given below, where $X = \{1, 2, 3\}$ and $\subseteq$ is the relation subset of.

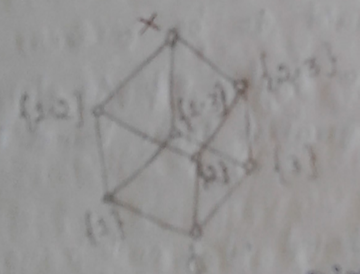


Figure 1: Hasse Diagram for $P(\{1,2,3\}, \subseteq)$

Definition: 1

A partial order relation $\leq$ on a set A is called a total order, or linear order, if for every $a, b \in A$ either $a \leq b$ or $b \leq a$. If $\leq$ is a total order on A, then the poset $(A, \leq)$ is called a chain, or a totally ordered set.

Remark: The poset given in example 2 is a chain. The poset $P(\{1,2,3\}, \subseteq)$ is not a chain.

Definition: 2

Let $(X, \leq)$ be a poset and $a, b \in X$. If there is an element c in X such that $a \leq c$ and $b \leq c$, then c is said to be an upper bound for a and b. An element c in X is said to be a least upper bound of a and b if

(i) c is an upper bound of a and b (i.e. $a \leq c$ & $b \leq c$) and

(ii) whenever d is an upper bound of a and b, then $c \leq d$ (i.e., $a \leq d$ and $b \leq d \Rightarrow c \leq d$)

An element c is said to be a lower bound of a and b if $c \leq a$ and $c \leq b$. An element c is said to be a greatest lower bound (g.l.b) of a and b if

(i) c is a lower bound of a and b; and

(ii) whenever d is a lower bound of a and b, then $d \leq c$.

Example:-

In the poset $(D(12), \leq)$, where $\leq$ is the relation 'is a divisor of' $2 \vee 3 = 6$, $3 \vee 6 = 6$, $4 \vee 6 = 12$; $2 \wedge 3 = 1$, $3 \wedge 6 = 3$, $4 \wedge 6 = 2$

2M

Definition : 3

A poset $(X, \leq)$ is said to be a lattice if for every $a, b \in X$, both $a \vee b$ and $a \wedge b$ exists.

Example: If X is a set, then the poset $(P(X), \subseteq)$ is a lattice.

Theorem: 1

Every chain is a lattice.

Proof:

Let $(X, \leq)$ be a chain and $a, b \in X$.

Then we have either $a \leq b$ or $b \leq a$.

Case 1: Assume that $a \leq b$.

clearly b is an upper bounded of a and b. we have If c is an upper bounded of a and b $a \leq c$ and $b \leq c$. Thus $b \leq c$, for every upper bound c of a and b.

Hence b is the least upper bound of a and b. That is $a \vee b = b$ similarly $a \wedge b = a$

Case 2: Assume that $b \leq a$. Then we can prove that $a \vee b = a$ and $a \wedge b = b$. Thus $a \vee b$, $a \wedge b$ exists for all $a, b \in X$ and hence $(X, \leq)$ is a lattice.

Definition: 4

Let $(X, \leq)$ be a poset. If there is an element $a \in X$ such that $a \leq x$, for all $x \in X$, then a is said to be a least element in X. An element $b \in X$ such that $x \leq b$, for all $x \in X$, is said to be a greatest element in X.

... is a least element in $(X, \leq)$, then

If there exists a greatest element, if it exists, is denoted by 1 and the least element, if it exists, is denoted by 0. A lattice which has both 0 and 1 is called a bounded lattice.

Example:

In the lattice $(P(X), \subseteq)$, where X is a se... the null set ϕ is the least element and the set X is the greatest element.

SOME PROPERTIES OF LATTICES

Theorem: 2

Let $(L, \leq)$ be a lattice, Then L satisfies the following laws.

1. Idempotent laws:

$$a \wedge a = a \text{ and } a \vee a = a \text{ for all } a \in L.$$

2. Commutative laws:

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a \text{ for all } a, b ...$$

3. Associative laws:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \text{ and}$$

$$(a \vee b) \vee c = a \vee (b \vee c) \text{ for all } a, b, c \in L$$

4. Absorption laws:

$$a \wedge (a \vee b) = a \text{ and } a \vee (a \wedge b) = a \text{ for all } a, b \in L.$$

proof:

Let $a \in L$. for all $x, y, z \in L$, by the defi... of glb of x and y, we have

$x \wedge y \leq x$ → ①

and if $z \leq x$ and $z \leq y$, then $z \leq x \wedge y$ → ②

Now, take $x = y = z = a$. As $a \leq a$ from ① and ② respectively, we have $a \wedge a \leq a$ and $a \leq a \wedge a$.

By the antisymmetric property, it follows that $a = a \wedge a$. Similarly we can prove that $a \vee a = a$.

2. Given a and $b \in L$, both $a \wedge b$ and $b \wedge a$ are glb's of a and b. By the uniqueness of glb of a and b, we have $a \wedge b = b \wedge a$.

Similarly $a \vee b = b \vee a$ holds good.

3. Let $a, b, c \in L$.

By the definition we have $(a \wedge b) \wedge c \leq (a \wedge b)$ and $(a \wedge b) \wedge c \leq c$. By the definition of glb of a and b, we have $a \wedge b \leq a$ and $a \wedge b \leq b$ so by transitive property of '$\leq$' we have $(a \wedge b) \wedge c \leq a$ and $(a \wedge b) \wedge c \leq b$.

As $(a \wedge b) \wedge c \leq b$ and $(a \wedge b) \wedge c \leq c$, we see that $(a \wedge b) \wedge c$ is a lower bound for b and c.

From the definition of $b \wedge c$, it follow that $(a \wedge b) \wedge c \leq b \wedge c$.

As $(a \wedge b) \wedge c \leq a$ and $(a \wedge b) \wedge c \leq (b \wedge c)$, from the definition of $a \wedge (b \wedge c)$, we have $(a \wedge b) \wedge c \leq a \wedge (b \wedge c) \rightarrow (3)$

Now, $a \wedge (b \wedge c) \leq a$ and $a \wedge (b \wedge c) \leq b \wedge c$.

As $b \wedge c \leq b$, by transitivity, $a \wedge (b \wedge c) \leq b$.

since $a \wedge (b \wedge c) \leq a$ and $a \wedge (b \wedge c) \leq b$,

we have $a \wedge (b \wedge c) \leq (a \wedge b)$

As $a \wedge (b \wedge c) \leq b \wedge c \leq c$

$$a \wedge (b \wedge c) \leq (a \wedge b) \wedge c \rightarrow (4)$$

From (3) and (4), by antisymmetric property,

It follows that $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Similarly we can prove that

$$a \vee (b \vee c) = (a \vee b) \vee c$$

4. Let $a, b \in L$. Then $a \leq a$ and $a \leq a \vee b$

so $a \leq a \wedge (a \vee b)$, on the other hand

$a \wedge (a \vee b) \leq a$. By antisymmetric property

of $\leq$, we have $a = a \wedge (a \vee b)$

Similarly, we have $a = a \vee (a \wedge b)$ $\forall a, b \in L$

Theorem : 3

Let $(L, \leq)$ be a lattice. For any $a, b \in L$,

the following are equivalent.

(i) $a \leq b$

(ii) $a \vee b = b$

(iii) $a \wedge b = a$.

Proof:

(i) $\Rightarrow$ (ii)

Assume that $a \leq b$.

As $a \leq b$ and $b \leq b$. From the definition of $a \vee b$, we have $a \vee b \leq b$. On the other hand $b \leq a \vee b$.

Hence by antisymmetric property of $\leq$,

we have $a \vee b = b$

(ii) $\Rightarrow$ (iii)

Assume that $a \vee b = b$.

Then $a \wedge b = a \wedge (a \vee b) = a$ (by absorption law)

(iii) $\Rightarrow$ (i)

Assume that $a \wedge b = a$. Then a is a lower bound for a and b. In particular $a \leq b$.

DUALITY PRINCIPLE:

2M (2) Let $(L, \leq)$ be a lattice. If we define a relation $\leq'$ in L as follows:

for all $a, b \in L$, $a \leq' b$ if and only if $b \leq a$ in $(L, \leq)$, then $\leq'$ is also a partial ordering on L. The partial ordering $\leq'$ is called the reversal of the partial ordering $\leq$.

Clearly for all $a, b \in L$.

lub $\{a, b\}$ in $(L, \leq') =$ glb $\{a, b\}$ in $(L, \leq)$

and glb $\{a, b\}$ in $(L, \leq') =$ lub $\{a, b\}$ in $(L, \leq)$

Therefore $(L, \leq')$ is also a lattice. This lattice is called the dual of the lattice $(L, \leq)$.

For example, in any lattice $(L, \leq)$, the statement

$a \leq b \Rightarrow a \vee b = b$ is valid.

Hence its dual statement $a \geq b \Rightarrow a \wedge b = b$ is valid.

Theorem : 4

In any lattice $(L, \leq)$, the operations $\vee$ and $\wedge$ are isotone. i.e, if $y \leq z$ in L. Then

$$x \wedge y \leq x \wedge z \text{ and } x \vee y \leq x \vee z \text{ for all } x \in L$$

proof:-

Let $x, y, z \in L$ and $y \leq z$

By Idempotent law, $x \wedge x = x$, As $y \leq z$, we have $y \wedge z$

So $x \wedge y = (x \wedge x) \wedge (y \wedge z) = x \wedge ((x \wedge y) \wedge z)$

$= x \wedge ((y \wedge x) \wedge z)$

$= x \wedge (y \wedge (x \wedge z))$

$= (x \wedge y) \wedge (x \wedge z)$.

using associative law.

From $x \wedge y = (x \wedge y) \wedge (x \wedge z)$, and by Theorem (3),

we have $x \wedge y \leq x \wedge z$

Thus if $y \leq z$, then $x \wedge y \leq x \wedge z$.

The dual of this statement is also true. so if $y \geq z$, then $x \vee y \geq x \vee z$, interchanging the role of y and z, in this statement, we get the following true statement If $z \geq y$, then

$$x \vee z \geq x \vee y$$

Theorem : 5

The elements of an arbitrary lattice $(L, \leq)$ satisfy the following inequalities:

1. Distributive Inequalities.

(i) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

(ii) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$

2. Modular Inequalities

(iii) $x \leq z \Rightarrow x \vee (y \wedge z) \leq (x \vee y) \wedge z$

(iv) $x \geq z \Rightarrow x \wedge (y \vee z) \geq (x \wedge z) \vee z$

proof:

As (ii) and (iv) are duals of (i) and (iii) respectively, It is enough to prove (i) and (iii) only.

First we prove (i).

Let $x, y, z \in L$. As $x \leq x \vee y$ and $x \leq x \vee z$, we have $x \leq (x \vee y) \wedge (x \vee z)$.

As $y \wedge z \leq y \leq x \vee y$ and $y \wedge z \leq z \leq x \vee z$, we have $(y \wedge z) \leq (x \vee y)$

$\Rightarrow (x \vee y) \wedge (x \vee z)$ is an upper bound for x and $y \wedge z$ and hence $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

Thus (i) is proved.

The Inequality (iii) is a special case of (i).

If $x \leq z$, then $x \vee z = z$ and so from (i), we obtain $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) = (x \vee y) \wedge z$

which is the Inequality (iii).

NEW (LATTICES (Definition)

A non-empty subset S of a lattice L is called a sublattice of L if S is closed under the operations 'join' and 'meet' of L. i.e., for all $s_1, s_2 \in S$, the glb $\{s_1, s_2\}$ and lub $\{s_1, s_2\}$ in the lattice L are elements of S.

Example:

Let $(L, \leq)$ be a lattice.

a) If $x \in L$, The set $\{x\}$ is a sublattice of L.

b) If $x, y \in L$ such that $x \leq y$, then the set $[x, y] = \{z \in L \mid x \leq z \leq y$ in $L\}$ is a sublattice of L.

LATTICE HOMOMORPHISMS :-

2M (3) Definition :- 2

Let $(L_1, \wedge, \vee)$ and $(L_2, \sqcap, \sqcup)$ be lattices and $f : L_1 \to L_2$ be a map.

Then f is said to be a

(i) meet - homomorphism if

$$f(x \wedge y) = f(x) \sqcap f(y), \text{ for all } x, y \in L_1$$

(ii) Join - homomorphism if

$$f(x \vee y) = f(x) \sqcup f(y), \text{ for all } x, y \in L_2$$

(iii) (lattice) homomorphism if it is both meet - homomorphism and join - homomorphism

(iv) order - preserving map if

$$x \leq y \text{ in } L_1 \Rightarrow f(x) \leq f(y) \text{ in } L_2$$

(v) order - reserving map if

$$\cancel{x \geq y}\ x \geq y \text{ in } L_1 \Rightarrow f(x) \geq f(y) \text{ in } L_1$$

Definition : 3.

Two posets $(P, \leq)$ and $(Q, \leq')$ are called order isomorphic if there is a bijective map $f : P \to Q$ such that $x \leq y$ in P if and only if $f(x) \leq' f(y)$ in Q.

Example : 4

Let L_1 and L_2 be the lattices represents by the Hasse diagrams given in the figure. Let $f_1, f_2, f_3 : L_1 \to L_2$ be the maps given by

$f_1(0) = 0$, $f_1(a) = f_1(b) = f_1(c) = f_1(d) = f_1(1) = b$;

$f_2(0) = 0$, $f_2(a) = f_2(b) = f_2(c) = b$, $f_2(d) = f_2(1) = 1$;

$f_3(0) = 0$, $f_3(a) = a$, $f_3(b) = f_3(c) = b$, $f_3(d) = f_3(1) = 1$.

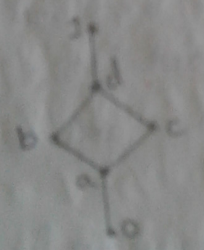
Then (i) f_1 is both meet and join-homomorphism

(ii) f_2 is a meet homomorphism but not join-homomorphism

(ie) ($f_2(b \vee c) = f_2(d) = 1$, but $f_2(b) \vee f_2(c) = b \neq 1$)

(iii) f_3 is neither a meet-homomorphism nor a join-homomorphism.

Note that all these three maps are order-preserving maps. The map f_3 is an example for an order-preserving map which is neither a meet-homomorphism nor a join-homomorphism.



Theorem : 1

Every meet-homomorphism (join-homomorphism) is an order preserving map.

Proof:

Let $L_1 \to L_2$ be a meet-homomorphism from a ... lattice $(L_2, \leq_2)$

Let $a \leq_1 b$ in L_1

Then $a \wedge b = a$

As f is a meet homomorphism, we have $f(a) = f(a \wedge b) = f(a) \wedge f(b)$

Thus $f(a) \wedge f(b) = f(a)$ in L_2.

Hence $f(a) \leq_2 f(b)$ in L_2

Thus $a \leq_1 b$ in $L_1 \Rightarrow f(a) \leq_2 f(b)$ in L_2 and f is an order-preserving map.

(The proof for join-homomorphism is similar).

Theorem: 8

Let f be an (order) isomorphism from a poset $(L, \leq)$ onto a poset $(M, \leq')$. If L is a lattice, then M is also a lattice and f is a lattice isomorphism.

Proof:

Let $f: L \to M$ be an order isomorphism. Assume L to be a lattice. Let $x, y \in M$. As f is a bijection, there exist a unique $a \in L$. (c exists as L is a lattice).

Let $z = f(c)$ in M. As $a \leq c$, $b \leq c$ and f is order preserving, we have $f(a) \leq' f(c)$ and $f(b) \leq' f(c)$ in M. i.e we have $x \leq' z$ and $y \leq' z$ in M and z is upper bound for $\{x, y\}$ in M.

Let $w \in M$ be an upper bound for $\{x, y\}$ in M. Let d be the unique element element in L such that $f(d) = w$. As $x \leq' w$, $y \leq' w$. (i.e $f(a) \leq' f(d)$, $f(b) \leq' f(d)$) and f is an order isomorphism, we have $a \leq' d$ and $b \leq' d$ in L. So, $c = a \vee b \leq d$ in L. As $c \leq d$, we have $f(c) \leq' f(d)$ in M. So we get $z \leq' w$ is M.

Thus we have proved that

(i) z is an upper bound for $\{x,y\}$ in M and

(ii) whenever w is an upper bound for $\{x,y\}$ in M, then $z \leq' w$.

In the other words, we have prove that $x \vee y$ exists in M and $x \vee y = z = f(c) = f(a \vee b)$, where $f(a) = x$ and $f(b) = y$.

Similarly, for all $x, y \in M$, the gcd $x \wedge y$ exists in M and if $x = f(a)$ and $y = f(b)$, then $x \wedge y = f(a \wedge b)$, so M is a lattice, and as the bijection $f: L \to M$ satisfies $f(a \vee b) = f(a) \vee f(b)$; $f(a \wedge b) = f(a) \wedge f(b)$, for all $a, b \in L$, f is a lattice isomorphism.

Product lattice of two lattices:

If L and M are lattices, then we can give a lattice structure to the cartesian product $L \times M$, using the partial order relations of L and M.

Let $(L, \wedge, \vee)$ and $(M, \sqcap, \sqcup)$ be two lattices. Consider the cartesian product $L \times M = \{(x,y) : x \in L$ and $y \in M\}$; we define binary operations $\wedge$ and $\vee$ as follows:

for all $(x_1, y_1), (x_2, y_2) \in L \times M$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \sqcap y_2)$$

and

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \sqcup y_2)$$

we now prove that $(L \times M, \wedge, \vee)$ is a lattice, using theorem 5 (of 2)

Theorem: 9

$(L \times M, \wedge, \vee)$ is a lattice.

proof:

we now show that $\wedge$ and $\vee$ defined above satisfy:

a) commutative laws

b) Associative laws

c) Absorption laws.

As $(L, \wedge, \vee)$ and $(M, \sqcap, \sqcup)$ are lattices, the operations $\wedge$ and $\vee$ on L and the operation $\sqcap$ and $\sqcup$ on M satisfy these laws.

Let (x_1, y_1), (x_2, y_2) and $(x_3, y_3) \in L \times M$

a) $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \sqcup y_2)$

$= (x_2 \vee x_1, y_2 \sqcup y_1)$

$= (x_2, y_2) \vee (x_1, y_1)$

similarly $(x_1, y_1) \wedge ((x_2, y_2) \wedge (x_3, y_3)) = ((x_1, y_1) \wedge (x_2, y_2)) \wedge (x_3, y_3)$

b) similarly $(x_1, y_1) \wedge (x_2, y_2) = (x_2, y_2) \wedge (x_1, y_1)$

b) $(x_1, y_1) \vee ((x_2, y_2) \vee (x_3, y_3)) = (x_1, y_1) \vee (x_2 \vee x_3, y_2 \sqcup y_3)$

$= (x_1 \vee (x_2 \vee x_3), y_1 \sqcup (y_2 \sqcup y_3))$

$= ((x_1 \vee x_2) \vee x_3, (y_1 \sqcup y_2) \sqcup y_3)$

$= (x_1 \vee x_2, y_1 \sqcup y_2) \vee (x_3, y_3)$

$= ((x_1, y_1) \vee (x_2, y_2)) \vee (x_3, y_3)$

similarly, $(x_1, y_1) \wedge ((x_2, y_2) \wedge (x_3, y_3))$

$= ((x_1, y_1) \wedge (x_2, y_2)) \wedge (x_3, y_3)$

c) $(x_1, y_1) \vee ((x_1, y_1) \wedge (x_2, y_2)) = (x_1, y_1) \vee (x_1 \wedge x_2, y_1 \sqcap y_2)$

$= (x_1 \vee (x_1 \wedge x_2), y_1 \vee (y_1 \sqcap y_2))$

$= (x_1, y_1)$

similarly, $(x_1, y_1) \vee ((x_1, y_1) \wedge (x_2, y_2)) = (x_1, y_1)$

Thus $(L \times M, \wedge, \vee)$ is a lattice

Definition :- 4

The lattice $(L \times M, \wedge, \vee)$ is called the product lattice of the lattice L and M.

Example :- 5

If L and M are lattices represented by Hasse diagrams given in figures 9(a) and 9(b) respectively, then the direct product $L \times M$ is represented by the Hasse diagram given in figure 9(c).

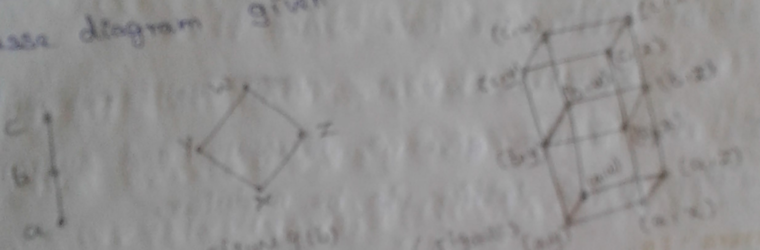


figure 9(a) lattice L

figure 9(b) lattice M

(figure 9(c) lattice L×M)

Modular and Distributive Lattices :-

we have already seen that in any lattice L, the modular inequalities

$x \le z \Rightarrow x \vee (y \wedge z) \le (x \vee y) \wedge z$

$x \ge z \Rightarrow x \wedge (y \vee z) \ge (x \wedge y) \vee z$

and the distributive inequalities

$x \vee (y \wedge z) \le (x \vee y) \wedge (x \vee z)$

$x \wedge (y \vee z) \ge (x \wedge y) \vee (x \wedge z)$

hold for $x, y, z \in L$.

But in general if $x \le z$, the inequality $x \vee (y \wedge z) \ge (x \vee y) \wedge z$ and if $x, y, z \in L$, the inequality $x \vee (y \wedge z) \ge (x \vee z) \wedge (x \vee z)$ need not hold.

Definition : 1

A lattice L is called modular if for all $x, y, z \in L$.

$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$ (modular equation)

Example : 3

The lattice N_5 is not modular.

In the discussion before the definition of a modular lattice we have shown that

$x \vee (y \wedge z) \neq (x \vee y) \wedge z$, Hence N_5 is not modular.

The next theorem gives a necessary and sufficient condition for a lattice to be modular.

Theorem : 10

A lattice L is modular if and only if none of a sublattice is isomorphic to the pentagon lattice N_5.

proof :

The pentagon lattice is not modular and hence any lattice having a pentagon as a sublattice cannot be modular. To prove the converse,

Let $(L, \leq)$ be a non-modular lattice.

As it is not modular, there are elements a, b, c in L such that $a \leq c$ and $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Let $u = b \wedge c$

$x = a \vee (b \wedge c)$

$y = b$

$z = (a \vee b) \wedge c$

and $v = a \vee b$

only can verify that these five elements are all distinct. we have $u \le x \le z \le v$ and $u \le y \le v$.

Therefore

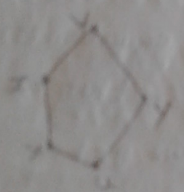
(i) $u \le x \wedge y \le z \wedge y = (a \vee b) \wedge c \wedge b = ((a \vee b) \wedge b) \wedge c = b \wedge c = a$

So $x \wedge z = z \wedge y = u$.

(ii) $v \ge z \vee y \ge x \vee y = a \vee (b \wedge c) \vee b = a \vee ((b \wedge c) \vee b) = a \vee b = v$

So $z \vee y = x \vee y = v$

These observations lead to the representation of $s = \{u, x, y, z, v\}$ by a Hasse diagram given in figure. So the sublattice $s = \{u, x, y, z, v\}$ is isomorphic to N_5.



Hasse diagram of $s = \{u, x, y, z, v\}$

Thus we have proved that if $(L, \le)$ is not modular, then it has a sublattice isomorphic to N_5.

Theorem 1.12

A lattice L is modula iff for all $x, y, z \in L$
$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$

proof :-

If L is modular, as $x \le (x \vee z)$, by modular equation we have $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$

Conversely, assume that for all $x, y, z \in L$

$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z) \rightarrow$ ②

If $x \leq z$, then $x \vee z = z$ and (*) becomes

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

which is the modular equation.

So the lattice L is modular.

Theorem : 13

A lattice L is modular iff for all $x, y, z \in L$

$$(x \vee (y \wedge z)) \wedge (y \vee z) = (x \wedge (y \vee z)) \vee (y \wedge z)$$

proof :

$$(x \vee (y \wedge z)) \wedge (y \vee z) = (x \wedge (y \vee z)) \vee (y \wedge z) \rightarrow (1)$$

If L is modular, as $y \leq z \leq y \leq z$, we have

$$(y \wedge z) \vee (x \wedge (y \vee z)) = ((y \wedge z) \vee x)(y \vee z)$$
$$= (x \vee (y \wedge z)) \wedge (y \vee z)$$

Conversely, assume that the equation (1) is valid for all $x, y, z \in L$. If $y \leq z$, then as

$y \wedge z = y$ and $y \vee z = z$, we get

$(x \vee (y \wedge z)) \wedge (y \vee z) = (x \vee y) \wedge z$ and

$(x \wedge (y \vee z)) \vee (y \wedge z) = (x \wedge z) \vee y$

So if $y \leq z$, $y \vee (x \wedge z) = (y \vee x) \wedge z$

Thus L is modular.

Definition : 2

A lattice L is called a distributive if either of the following conditions hold for all x, y, z in L.

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

(This is called the distributive eqn)

Theorem : 14

5M 3(a) Every chain is a distributive lattice.

proof:

Let $(L, \leq)$ be a chain and $a, b, c \in L$.

Consider the following possible cases.

(i) $a \leq b$ or $a \leq c$ and

(ii) $b \leq a$ and $c \leq a$

In case (i), $a \leq b \vee c$, so $a \wedge (b \vee c) = a$

and $(a \wedge b) \vee (a \wedge c) = a$

In case (ii), $b \vee c \leq a$, so $a \wedge (b \vee c) = b \vee c$

and $(a \wedge b) \vee (a \wedge c) = b \wedge c$

Thus for all $a, b, c \in L$, the distributive equation

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

holds and hence L is distributive.

3 (b) Theorem : 15

Every distributive lattice is modular.

proof:

Let $(L, \leq)$ be a distributive lattice.

For all $x, y, z \in L$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Thus if $x \leq z$, then $x \vee z = z$ and $x \vee (y \wedge z) = (x \vee y) \wedge z$,

So if $x \leq z$, the modular equation is satisfied

and L is modular.

Theorem : 17

A modular lattice is distributive iff none of its sublattices is isomorphic to the 'diamond lattice' M_5.

proof :-

As the diamond lattice M_5 is not distributive, any lattice having a sublattice isomorphic to M_5 cannot be distributive.

Conversely, let L be a modular lattice but not distributive.

So, by theorem 16, we can find x, y, z in L such that $(x\wedge y)\vee(y\wedge z)\vee(z\wedge x) < (x\vee y)\wedge(y\vee z)\wedge(z\vee x)$ → (*)

Let $u = (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$

$v = (x\vee y)\wedge(y\vee z)\wedge(z\vee x)$

$a = u\vee(x\wedge v)$

$b = u\vee(y\wedge v)$ and

$c = u\vee(z\wedge v)$

One can verify that these 5 elements u, v, a, b, c from a sublattice which is isomorphic to the diamond lattice M_5.

Theorem : 18

Let L be a distributive lattice and $a, b, c \in L$. If $a\wedge b = a\wedge c$ and $a\vee b = a\vee c$, then $b = c$.

(This result known as 'Cancellation Rule' for D.L.)

proof:

Let $a, b, c \in L$, such that $a \wedge b = a \wedge c$ & $a \vee b = a \vee c$

Now, $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, by the distributive property

$= (a \vee b) \wedge (b \vee c)$ as $a \vee c = a \vee b$

$= (b \vee a) \wedge (b \vee c)$ as $a \vee b = b \vee a$

$= b \vee (a \wedge c)$ distributive property

$= b \vee (a \wedge b)$ as $a \wedge c = a \wedge b$

$= b$ by absorption law

Also $(a \wedge b) \vee c = (a \wedge c) \vee c$ as $a \wedge b = a \wedge c$

$= c$ by absorption law

Thus $b = (a \wedge b) \vee c = c$

So $a \wedge b = a \wedge c$ and $a \vee b = a \vee c \Rightarrow b = c$.

UNIT - V

2M (5) **Definition : 1** (Boolean Algebras)

A complemented distributive lattice is called a Boolean algebra or a Boolean lattice.

Example : 1

Let x be a set. Then the lattice $p(x)$, the set of all subset of x, under the usual set theoretic inclusion as partial ordering is a Boolean algebra. If $A, B \in p(x)$, then

$A \wedge B = A \cap B$;

$A \vee B = A \cup B$; $\phi \leq A \leq x$, $A \cup (x - A) = x$

and $A \cap (x - A) = \phi$. so $p(x)$ is a lattice with $0 = \phi$, $1 = x$, and to each $A \in p(x)$, its complement is unique and it is $A - x$. Also distributive lattice in otherwords, it is a Boolean algebra.

In fact we show that every finite Boolean algebra is isomorphic to some (power set) Boolean Algebra $p(A)$.

Theorem : 2.1

In a Boolean algebra L, the De Morgan's laws, given by $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$ holds for all $a, b \in L$.

proof :

In a Boolean algebra L, the De Morgan's

Let L be a Boolean algebra and $a, b \in L$.

Then $(a \wedge b) \vee (a' \vee b') = (a \vee (a' \vee b)) \wedge (b \vee (a' \vee b'))$ by distributive law

$= ((a \vee a') \vee b') \wedge ((b \vee b') \vee a')$

$= (1 \vee b') \wedge (1 \vee a')$

$= 1 \wedge 1$ as $a' \leq 1$ and $b' \leq 1$

$= 1$

and $(a \wedge b) \wedge (a' \vee b') = ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b')$ by distributive law

$= ((a \wedge a') \wedge b) \vee (a \wedge (b \wedge b'))$

$= (0 \wedge b) \vee (a \wedge 0)$

$= 0 \vee 0 = 0$

Thus $a' \vee b'$ is the complement of $a \wedge b$.

i.e) $(a \wedge b)' = a' \vee b'$

$(a \vee b)' = a' \wedge b'$ follows by applying the duality principle to $(a \wedge b)' = a' \vee b'$

Definition: 5

Let a and b are two elements in a lattice. The element b is said to be a cover for a if $a < b$ (i.e $a \neq b$

Theorem: 2.2

Let B be a finite Boolean algebra. If $b \neq 0$ is an element in B, then there exist an atom a such that $a \leq b$.

proof :-

If b itself is an atom, then we take $a = b$. If b is not an atom, as B is finite, we can find a chain $0 < b_1 < b_2 < \cdots < b_n < b$ satisfying $0 < b_1$; $b_i < b_{i+1}$ for all $i = 1, 2, \ldots, n-1$ and $b_n < b$. So b_1 is an atom such that $b_1 \leq b$ and we take $a = b_1$.

Theorem : 23

Let B be a finite Boolean algebra and $b \neq 0$ in B. Let $a_1, a_2, \ldots, a_k$ be all the atoms of B such that $a_i \leq b$. Then $b = a_1 \vee a_2 \vee \cdots \vee a_k$

proof :-

Let $b \neq 0$ in B. Define $A(b) = \{a \in B : 0 < a \,\&\, a \leq b\}$

By theorem 22, $A(b) \neq \phi$.

As B itself is finite, $A(b)$ is a finite set

Let $A(b) = \{a_1, a_2, \ldots, a_k\}$ and $c = a_1 \vee a_2 \vee \cdots \vee a_k$.

As each $a_i \leq b$, we have $c \leq b$.

We claim that $b \leq c$

It is enough the show that $b \wedge c' = 0$.

If $b \wedge c' \neq 0$, then $A(b \wedge c') \neq \phi$. Consider an atom a of B such that $a \leq b \wedge c'$. Then $a \leq c'$ and $a \leq b$.

As $a \leq b$ and a is an atom, $a \in A(b)$.

So $a = a_i$ for some i and $a \leq c$. As $a \leq c$ and $a \leq c'$, we have $a \leq c \wedge c' = 0$. which is a contradiction to $a \neq 0$ (by the defn of atom).

Thus $b \wedge c' = 0$

and hence $b \leq c$ so $b = a_1 \vee a_2 \vee \cdots \vee a_k$.

Theorem : 24

Let B be a finite Boolean algebra and $b \neq 0$ in B. If $a_1, a_2, \ldots, a_k$; $b_1, b_2, \ldots, b_m$ are atoms of B such that

(i) $a_1, a_2, \ldots, a_k$ are distinct

(ii) $b_1, b_2, \ldots, b_m$ are distinct and

(iii) $b = a_1 \vee a_2 \vee \cdots \vee a_k = b_1 \vee b_2 \vee \cdots \vee b_m$

Then $k = m$ and $\{a_1, a_2, \ldots, a_k\} = \{b_1, b_2, \ldots, b_m\}$

proof :-

By (i), (ii) and (iii) $a_1, a_2, \ldots, a_k$ are distinct atoms of B and $b_1, b_2, \ldots, b_m$ are distinct atoms of B such that $b = a_1 \vee a_2 \vee \cdots \vee a_k = b_1 \vee b_2 \vee \cdots \vee b_m$

Then each $a_i \leq b$ and each $b_i \leq b$

$$\text{so } a_i = a_i \wedge b = a_i \wedge (b_1 \vee b_2 \vee \cdots \vee b_m)$$
$$= (a_i \wedge b_1) \vee (a_i \wedge b_2) \vee \cdots \vee (a_i \wedge b_m)$$

As $a_i \neq 0$, we can find j such that $a_i \wedge b_j \neq 0$.

As both a_i and b_j are atoms and $a_i \wedge b_j \neq 0$, $a_i = b_j$. Thus each a_i is the same as some b_j.

Similarly, $b_s = b_s \wedge b = b_s \wedge (a_1 \vee a_2 \vee \cdots \vee a_k)$
$$= (b_s \wedge a_1) \vee (b_s \wedge a_2) \vee \cdots \vee (b_s \wedge a_k)$$

As $b_s \neq 0$, $b_s \wedge a_i \neq 0$ for some a_i. As b_s and a_i are atoms $b_s = a_i$. From this we have $\{a_1, a_2, \ldots a_k\} = \{b_1, b_2, \ldots b_m\}$ and as $a_1, a_2, \ldots, a_k$ are distinct and $b_1, b_2, \ldots, b_m$ are distinct, we also have $k = m$.

By theorem 2.3 and 2.4, we have shown that every non-zero element of a finite Boolean algebra can be expressed as a join of atoms and this expression is unique. In fact

$b = a_1 \vee a_2 \vee \dots a_k$ where $\{a_1, a_2, \dots a_k\}$ is the set of all atoms of B that are $\leq b$.

3m Definition 1.3

Let B_1 and B_2 be Boolean algebras. A mapping $f: B_1 \to B_2$ is called a Boolean homomorphism from B_1 into B_2 if f is a lattice homomorphism and for all $x \in B$, we have $f(x') = (f(x))^*$.

A Boolean homomorphism is said to be an isomorphism, if it is bijective. If there is a Boolean isomorphism between B_1 and B_2, we write

$B_1 \cong_b B_2$.

Now, we have the following representation theorem for finite Boolean algebras.

Theorem 2.5

Let B be a finite Boolean algebra and let A be the set of all atoms of B. Then the Boolean algebra B is isomorphic to the Boolean algebra P(A).

proof :-

To each $b \neq 0$ in B let $A(b) = \{a \in B ; 0 < a \leq b\}$ and let $A(0) = \phi$.

Then for all $b \neq 0$, $A(b) \neq \phi$ and $A(b) \subseteq A$, where $A = A(1)$ is the set of all atoms in B (as B is finite set). Now we define a map $h : B \to P(A)$ given by $h(b) = A(b)$ for all $b \in B$. we now show that the map is a Boolean isomorphism, by observing the following

Boolean Polynomials : Definition

Let $X_n = \{x_1, x_2, \dots, x_n\}$ be a set of n symbols, called indeterminates or variables, which do not contain the symbols 0 and 1. A Boolean polynomial (or a Boolean expression or a formula) in the variables x_k is defined recursively as follows:

1. The symbols 0 and 1 are Boolean polynomials.

2. $x_1, x_2, \dots, x_n$ are Boolean polynomials.

3. If $p = p(x_1, x_2, \dots, x_n)$ and $q = (x_1, x_2, \dots, x_n)$ are two Boolean polynomials, then so are

$$(p) \vee (q) \text{ and } (p) \wedge (q)$$

4. If $p = p(x_1, x_2, \dots, x_n)$ is a Boolean polynomial then so is $(p)'$. [By tradition $(0)', (1)', (x_k)'$ are denoted by $0', 1', x_k'$ respectively.]

5. There are no Boolean polynomials in the variables x_k, $k = 1, 2, \dots, n$ other than those that can be obtained by finitely many successive applications of rules 1, 2, 3 and 4.

Example :-

The following are some Boolean polynomials

$p_1(x_1, x_2, x_3) = x_1 \vee (x_2 \wedge x_3)$

$p_2(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge (x_1 \wedge x_2)'$

$p_3(x_1, x_2, x_3) = ((0 \vee x_1) \wedge (1 \wedge x_2)) \vee (x_2 \wedge x_3)'$

5M Theorem 1.36

1 Let B be the Boolean algebra $\{0, 1\}$ with the usual $\wedge, \vee, '$ operations, and B be any given finite Boolean algebra. Let $p, q \in P_n$. If $p_B = q_B$. Then $p_B = q_B$.

proof:-

Let B be a finite Boolean algebra.

Then $B = P(A)$, the power set of some finite set A. Each element C in $P(A)$ can be identified with the characteristic function $\chi_C : A \to \{0, 1\}$

[Now that $\chi_C(a) = 1$ iff $a \in C$]

Let $p, q \in P_n$ such that $p_B = q_B$. (i.e $p_B(a_1, a_2, \ldots, a_n)$ $= q_B(a_1, a_2, \ldots, a_n)$ for all $(a_1, a_2, \ldots, a_n) \in B^n$.

we claim then $p_B = q_B$ (where $B = P(A)$).

Let $(f_1, f_2, \ldots, f_n) \in (P(A))$. Note that each $f_i \in P(A)$ and so it is a map from A to $B = \{0, 1\}$. for all $a \in A$, we have

$p_B(f_1, f_2, \ldots, f_n)(a) = p_B(f_1(a), f_2(a), \ldots, f_n(a))$

$= q_B(f_1(a), f_2(a), \ldots, f_n(a))$

$= q_B(f_1, f_2, \ldots, f_n)(a)$

Thus $p_B(f_1, f_2, \ldots, f_n) = q_B(f_1, f_2, \ldots, f_n)$.

This is true for all $(f_1, f_2, \ldots, f_n) \in (P(A))^B$

Hence $\boxed{p_B = q_B}$

Definition : 2

Let $p, q \in P_n$ be Boolean polynomials in n variables. We say that $p \sim q$ if and only if $p_B = q_B$, where B in the Boolean algebra $\{0, 1\}$.

Definition : 3

If a Boolean polynomials p in n-variables $x_1, x_2, \ldots, x_n$ is of the form $x_1^{a_1} \wedge x_2^{a_2} \wedge \ldots \wedge x_n^{a_n}$ where each $a_i \in \{0, 1\}$ and $x_i^{a_i}$ stands for x_i' if $a_i = 0$ or stands for x_i if $a_i = 1$, then p is said to be a minimum minterm or a complete product or a fundamental product of the n-variables.

Example : 2

The following Boolean polynomials are minterms in 4 variables

$x_1 \wedge x_2' \wedge x_3' \wedge x_4$; $x_1' \wedge x_2 \wedge x_3 \wedge x_4$

Now, we record certain observations as remarks. These remarks lead to an idea of normal form of a Boolean polynomial.

Definition : 4

Let $p \neq 0$ be a Boolean polynomial in n-variables. The unique sum of minterms which is equivalent to p is said to be the sum-of-products canonical form for p, or the principal disjunctive normal form of p, and is denoted

Example: 7

Express the polynomial $p(x_1, x_2, x_3) = x_1 \vee x_2$ in an equivalent sum-of-products canonical form in three variables x_1, x_2 and x_3

Solution:-

$$x_1 \wedge x_2 = (x_1 \wedge (x_2 \vee x_2')) \vee (x_2 \wedge (x_1 \vee x_1'))$$

$$= (x_1 \wedge x_2) \vee (x_1 \wedge x_2') \vee (x_2 \wedge x_1) \vee (x_2 \wedge x_1')$$

$$= (x_1 \wedge x_2) \vee (x_1 \wedge x_2') \vee (x_1' \wedge x_2)$$

$$= ((x_1 \wedge x_2) \wedge (x_3 \vee x_3')) \vee ((x_1 \wedge x_2') \wedge (x_3 \vee x_3')) \vee ((x_1' \wedge x_2) \wedge (x_3 \vee x_3'))$$

$$= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3') \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3')$$

$$= min_7 \vee min_6 \vee min_5 \vee min_4 \vee min_3 \vee min_2$$

$$= m_7 \vee m_6 \vee m_5 \vee m_4 \vee m_3 \vee m_2$$

[This can be formally denoted by ⊕ 2,3,4,5,6,7 or by Σ 2,3,4,5,6,7].

[Note: The decimal numbers corresponding to binary numbers 111, 110, 101, 100, 011, 010 are 7, 6, 5, 4, 3 & 2 respectively].

Definition:-

If a Boolean polynomial p in n-variables $x_1, x_2, \ldots, x_n$ is of the form $x_1^{a_1} \vee x_2^{a_2} \vee \ldots \vee x_n^{a_n}$ where each $a_i \in \{0, 1\}$, then p is said to be a maxterm or a complete sum or a fundamental sum of the n-variable.

Example : 4

Express the polynomial $p(x_1, x_2, x_3) = x_1 \vee x_2$ in a equivalent products-of-sums canonical form in three variables x_1, x_2 and x_3

Solution :-

$$x_1 \vee x_2 = x_1 \vee x_2 \vee (x_3 \wedge x_3')$$
$$= (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3')$$

Problems

1. Find the principal disjunctive normal form of

$$p(x_1, x_2, x_3) = (x_2 + x_1 x_3)\overline{((x_1 + x_3) x_2)}$$

Solution :-

$$(x_2 + (x_1 x_3))\overline{((x_1 + x_3) x_2)}$$
$$= (x_2 + x_1 x_3)(\bar{x}_1 \bar{x}_3 + \bar{x}_2)$$
$$= x_2 \bar{x}_1 \bar{x}_3 + x_2 \bar{x}_2 + x_1 x_3 \bar{x}_1 \bar{x}_3 + x_1 x_3 \bar{x}_2$$
$$= x_2 \bar{x}_1 \bar{x}_3 + x_1 x_3 \bar{x}_2 \text{ as } x_2 \bar{x}_2 = x_1 \bar{x}_1 = 0.$$
$$= \bar{x}_1 x_2 \bar{x}_3 + x_1 \bar{x}_2 x_3$$

So $\bar{x}_1 x_2 \bar{x}_3 + x_1 \bar{x}_2 x_3$ is the principal disjunctive normal form of the given expression.

2. Expand the following function into the canonical sum-of-products forms $f(x, y, z) = xy + y\bar{z}$

Solution :-

$$f(x, y, z) = xy + y\bar{z}$$
$$= xy(z + \bar{z}) + (x + \bar{x}) y \bar{z}$$
$$= xyz + xy\bar{z} + xy\bar{z} + \bar{x} y \bar{z}$$

3. Write down the minterm normal of $f(x_1, x_2) = \bar{x}_1 \vee \overline{x_2}$

Solution :-

$$f(x_1, x_2) = \overline{x_1} \vee \overline{x_2}$$

$$= (\overline{x_1} \wedge (x_2 \vee \overline{x_2})) \vee ((x_1 \vee \overline{x_1}) \wedge \overline{x_2})$$

$$= (\overline{x_1} \wedge x_2) \vee (\overline{x_1} \wedge \overline{x_2}) \vee (x_1 \wedge \overline{x_2}) \vee (\overline{x_1} \wedge \overline{x_2})$$

$$= (\overline{x_1} \wedge x_2) \vee (x_1 \wedge \overline{x_2} \vee (\overline{x_1} \wedge \overline{x_2})$$